

A classification theorem and a spectral sequence for a locally free sheaf cohomology of a supermanifold

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To our team coach Yu.A. Kirillov on his 70th birthday

Abstract. This paper is based on the paper [3], where two classification theorems for locally free sheaves on supermanifolds were proved and a spectral sequence for a locally free sheaf of modules \mathcal{E} was obtained. We consider another filtration of the locally free sheaf \mathcal{E} , the corresponding classification theorem and the spectral sequence, which is more convenient in some cases. The methods, which we are using here, are similar to [2, 3].

The first spectral sequence of this kind was constructed by A.L. Onishchik in [2] for the tangent sheaf of a supermanifold. However, the spectral sequence considered in this paper is not a generalization of Onishchik's spectral sequence from [2].

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1. Main definitions and classification theorems

1.1. Main definitions

Let (M, \mathcal{O}) be a supermanifold of dimension $n|m$, i.e. a \mathbb{Z}_2 -graded ringed space that is locally isomorphic to a superdomain in $\mathbb{C}^{n|m}$. The underlying complex manifold (M, \mathcal{F}) is called the *reduction* of (M, \mathcal{O}) . The simplest class of supermanifolds constitute the so-called *split supermanifolds*. We recall that a supermanifold (M, \mathcal{O}) is called split if $\mathcal{O} \simeq \bigwedge_{\mathcal{F}} \mathcal{G}$, where \mathcal{G} is a locally free sheaf of \mathcal{F} -modules on M . With any supermanifold (M, \mathcal{O}) one can associate a split supermanifold $(M, \tilde{\mathcal{O}})$ of the same dimension which is called

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the *retract* of (M, \mathcal{O}) . To construct it, let us consider the \mathbb{Z}_2 -graded sheaf of ideals $\mathcal{J} = \mathcal{J}_0 \oplus \mathcal{J}_1 \subset \mathcal{O}$ generated by odd elements of \mathcal{O} . The structure sheaf of the retract is defined by

$$\tilde{\mathcal{O}} = \bigoplus_{p \geq 0} \tilde{\mathcal{O}}_p, \quad \text{where } \tilde{\mathcal{O}}_p = \mathcal{J}^p / \mathcal{J}^{p+1}, \quad \mathcal{J}^0 := \mathcal{O}.$$

Here $\tilde{\mathcal{O}}_1$ is a locally free sheaf of \mathcal{F} -modules on M and $\tilde{\mathcal{O}}_p = \bigwedge_{\mathcal{F}}^p \tilde{\mathcal{O}}_1$. By definition, the following sequences

$$\begin{aligned} 0 \rightarrow \mathcal{J} \cap \mathcal{O}_0 \rightarrow \mathcal{O}_0 \xrightarrow{\pi} \tilde{\mathcal{O}}_0 \rightarrow 0, \\ 0 \rightarrow \mathcal{J}^2 \cap \mathcal{O}_1 \rightarrow \mathcal{O}_1 \xrightarrow{\tau} \tilde{\mathcal{O}}_1 \rightarrow 0. \end{aligned} \quad (1.1)$$

are exact. Moreover, they are locally split. The supermanifold (M, \mathcal{O}) is split iff both sequences are globally split.

Denote by \mathcal{S}_0 and \mathcal{S}_1 the even and the odd parts of a \mathbb{Z}_2 -graded sheaf of \mathcal{O} -modules \mathcal{S} on M , respectively; by $\Pi(\mathcal{S})$ we denote the same sheaf of \mathcal{O} -modules \mathcal{S} equipped with the following \mathbb{Z}_2 -grading: $\Pi(\mathcal{S})_0 = \mathcal{S}_1$, $\Pi(\mathcal{S})_1 = \mathcal{S}_0$. A \mathbb{Z}_2 -graded sheaf of \mathcal{O} -modules on M is called *free (locally free) of rank $p|q$* , $p, q \geq 0$ if it is isomorphic (respectively, locally isomorphic) to the \mathbb{Z}_2 -graded sheaf of \mathcal{O} -modules $\mathcal{O}^p \oplus \Pi(\mathcal{O})^q$. For example, the tangent sheaf \mathcal{T} of a supermanifold (M, \mathcal{O}) is a locally free sheaf of \mathcal{O} -modules.

Let now $\mathcal{E} = \mathcal{E}_0 \oplus \mathcal{E}_1$ be a locally free sheaf of \mathcal{O} -modules of rang $p|q$ on an arbitrary supermanifold (M, \mathcal{O}) . We are going to construct a locally free sheaf of the same rank on $(M, \tilde{\mathcal{O}})$. First, we note that $\mathcal{E}_{\text{red}} := \mathcal{E} / \mathcal{J}\mathcal{E}$ is a locally free sheaf of \mathcal{F} -modules on M . Moreover, \mathcal{E}_{red} admits the \mathbb{Z}_2 -grading $\mathcal{E}_{\text{red}} = (\mathcal{E}_{\text{red}})_0 \oplus (\mathcal{E}_{\text{red}})_1$, by two locally free sheaves of \mathcal{F} -modules

$$(\mathcal{E}_{\text{red}})_0 := \mathcal{E}_0 / \mathcal{J}\mathcal{E} \cap \mathcal{E}_0 \quad \text{and} \quad (\mathcal{E}_{\text{red}})_1 := \mathcal{E}_1 / \mathcal{J}\mathcal{E} \cap \mathcal{E}_1$$

of ranks p and q , respectively. Further, the sheaf \mathcal{E} possesses the filtration

$$\mathcal{E} = \mathcal{E}_{(0)} \supset \mathcal{E}_{(1)} \supset \mathcal{E}_{(2)} \supset \dots, \quad \text{where } \mathcal{E}_{(p)} = \mathcal{J}^p \mathcal{E}_0 + \mathcal{J}^{p-1} \mathcal{E}_1, \quad p \geq 1. \quad (1.2)$$

Using this filtration, we can construct the following locally free sheaf of $\tilde{\mathcal{O}}$ -modules on M :

$$\tilde{\mathcal{E}} = \bigoplus_p \tilde{\mathcal{E}}_p, \quad \text{where } \tilde{\mathcal{E}}_p = \mathcal{E}_{(p)} / \mathcal{E}_{(p+1)}.$$

The sheaf $\tilde{\mathcal{E}}$ is also a locally free sheaf of \mathcal{F} -modules. In other words, $\tilde{\mathcal{E}}$ is a sheaf of sections of a certain vector bundle. The following exact sequence gives a description of $\tilde{\mathcal{E}}$.

$$0 \rightarrow \tilde{\mathcal{O}}_p \otimes (\mathcal{E}_{\text{red}})_0 \rightarrow \tilde{\mathcal{E}}_p \rightarrow \tilde{\mathcal{O}}_{p-1} \otimes (\mathcal{E}_{\text{red}})_1 \rightarrow 0.$$

We also have the following two exact sequences, which are locally split:

$$\begin{aligned} 0 \rightarrow \mathcal{E}_{(1)0} \rightarrow \mathcal{E}_{(0)0} \xrightarrow{\alpha} \tilde{\mathcal{E}}_0 \rightarrow 0; \\ 0 \rightarrow \mathcal{E}_{(2)1} \rightarrow \mathcal{E}_{(1)1} \xrightarrow{\beta} \tilde{\mathcal{E}}_1 \rightarrow 0. \end{aligned} \quad (1.3)$$

The sheaf $\tilde{\mathcal{E}}$ is \mathbb{Z} -graded by definition. Unlike the \mathbb{Z}_2 -grading considered in [3], the natural \mathbb{Z}_2 -grading is compatible with this \mathbb{Z} -grading.

$$(\tilde{\mathcal{E}})_{\bar{0}} := \bigoplus_{p=2k} \tilde{\mathcal{E}}_p, \quad (\tilde{\mathcal{E}})_{\bar{1}} := \bigoplus_{p=2k+1} \tilde{\mathcal{E}}_p.$$

1.2. Classification theorem for locally free sheaves \mathcal{E} on supermanifolds with given $\tilde{\mathcal{E}}$

Our objective now is to classify locally free sheaves \mathcal{E} of \mathcal{O} -modules on supermanifolds (M, \mathcal{O}) which have the fixed retract $(M, \tilde{\mathcal{O}})$ and such that the corresponding locally free sheaf $\tilde{\mathcal{E}}$ is fixed.

Let (M, \mathcal{O}) and (M, \mathcal{O}') be two supermanifolds, $\mathcal{E}, \mathcal{E}'$ be locally free sheaves of \mathcal{O} -modules and \mathcal{O}' -modules on M , respectively. Suppose that $\Psi : \mathcal{O} \rightarrow \mathcal{O}'$ is a superalgebra sheaf morphism. A vector space sheaf morphism $\Phi_\Psi : \mathcal{E} \rightarrow \mathcal{E}'$ is called a *quasi-morphism* if

$$\Phi_\Psi(fv) = \Psi(f)\Phi_\Psi(v), \quad f \in \mathcal{O}, \quad v \in \mathcal{E}.$$

As usual, we assume that $\Phi_\Psi(\mathcal{E}_{\bar{i}}) \subset \mathcal{E}'_{\bar{i}}, \bar{i} \in \{\bar{0}, \bar{1}\}$. An invertible quasi-morphism is called a *quasi-isomorphism*. A quasi-isomorphism $\Phi_\Psi : \mathcal{E} \rightarrow \mathcal{E}'$ is also called a *quasi-automorphism* of \mathcal{E} . Denote by $\mathcal{A}ut\mathcal{E}$ the sheaf of quasi-automorphisms of \mathcal{E} . It has a double filtration by the subsheaves

$$\mathcal{A}ut_{(p)(q)}\mathcal{E} := \{\Phi_\Psi \in \mathcal{A}ut\mathcal{E} \mid \Phi_\Psi(v) \equiv v \pmod{\mathcal{E}_{(p)}}, \Psi(f) = f \pmod{\mathcal{J}^q} \text{ for } v \in \mathcal{E}, f \in \mathcal{O}\}, \quad p, q \geq 0.$$

We also define the following subsheaf of $\mathcal{A}ut\tilde{\mathcal{E}}$:

$$\widetilde{\mathcal{A}ut}\tilde{\mathcal{E}} := \{\Phi_\Psi \mid \Phi_\Psi \in \mathcal{A}ut(\tilde{\mathcal{E}}), \Phi_\Psi \text{ preserves the } \mathbb{Z}\text{-grading of } \tilde{\mathcal{E}}\}. \quad (1.4)$$

If $\Phi_\Psi \in \widetilde{\mathcal{A}ut}\tilde{\mathcal{E}}$, then $\Psi : \tilde{\mathcal{O}} \rightarrow \tilde{\mathcal{O}}$ also preserves the \mathbb{Z} -grading. The 0-th cohomology group $H^0(M, \widetilde{\mathcal{A}ut}\tilde{\mathcal{E}})$ acts on the sheaf $\mathcal{A}ut\tilde{\mathcal{E}}$ by the automorphisms $\delta \mapsto a \circ \delta \circ a^{-1}$, where $a \in H^0(M, \widetilde{\mathcal{A}ut}\tilde{\mathcal{E}})$ and $\delta \in \mathcal{A}ut\tilde{\mathcal{E}}$. It is easy to see that this action leaves invariant the subsheaves $\mathcal{A}ut_{(p)(q)}\tilde{\mathcal{E}}$ and hence induces an action of $H^0(M, \widetilde{\mathcal{A}ut}\tilde{\mathcal{E}})$ on the cohomology set $H^1(M, \mathcal{A}ut_{(p)(q)}\tilde{\mathcal{E}})$. The unit element $\epsilon \in H^1(M, \mathcal{A}ut_{(p)(q)}\mathcal{E}')$ is a fixed point with respect to the action of $H^0(M, \mathcal{A}ut\mathcal{E}')$.

Let \mathcal{E} be a locally free sheaf of \mathcal{O} -modules on M . Denote

$$[\mathcal{E}] = \{\mathcal{E}' \mid \mathcal{E}' \text{ is quasi-isomorphic to } \mathcal{E}\}.$$

The total space of the bundle corresponding to a locally free sheaf \mathcal{E} will be denoted \mathbb{E} . It is a supermanifold. The locally free sheaf $\tilde{\mathcal{E}}$ corresponding to \mathcal{E} has the following property: The retract $\tilde{\mathbb{E}}$ of \mathbb{E} is the total space of the bundle corresponding to $\tilde{\mathcal{E}}$.

Theorem 1.1. *Let (M, \mathcal{O}') be a split supermanifold and \mathcal{E}' be a locally free sheaf of \mathcal{O}' -modules on M such that $\mathcal{E}' \simeq \tilde{\mathcal{E}}'$. Then*

$$\{[\mathcal{E}] \mid \tilde{\mathcal{O}} = \mathcal{O}', \tilde{\mathcal{E}} = \mathcal{E}'\} \xleftarrow{1:1} H^1(M, \mathcal{A}ut_{(2)(2)}\mathcal{E}')/H^0(M, \widetilde{\mathcal{A}ut}\mathcal{E}').$$

The orbit of the unit element ϵ , which is ϵ itself, corresponds to \mathcal{E}' .

Proof. Let \mathcal{E} be a locally free sheaf of \mathcal{O} -modules on (M, \mathcal{O}) and $\mathcal{U} = \{U_i\}$ be an open covering of M such that (1.1) and (1.3) are split (hence exact) over U_i and $\mathcal{E}|_{U_i}$ are free. In this case, $\tilde{\mathcal{E}}|_{U_i}$ are free sheaves of $\tilde{\mathcal{O}}$ -modules. We fix homogeneous bases (even and odd, respectively) (\hat{e}_j^i) and (\hat{f}_j^i) of the free sheaves of $\tilde{\mathcal{O}}$ -modules $\tilde{\mathcal{E}}|_{U_i}$, $U_i \in \mathcal{U}$. Without loss of generality, we may assume that $\hat{e}_j^i \in \tilde{\mathcal{E}}_0$ and $\hat{f}_j^i \in \tilde{\mathcal{E}}_1$. We are going to define an isomorphism $\delta_i : \mathcal{E}|_{U_i} \rightarrow \tilde{\mathcal{E}}|_{U_i}$.

Let $e_j^i \in \mathcal{E}_{(0)\bar{0}}$ be such that $\alpha(e_j^i) = \hat{e}_j^i$ and $f_j^i \in \mathcal{E}_{(0)\bar{1}}$ be such that $\beta(f_j^i) = \hat{f}_j^i$, see (1.3). Then (e_j^i, f_j^i) is a local basis of $\mathcal{E}|_{U_i}$. A splitting of (1.1) determines a local isomorphism $\sigma_i : \mathcal{O}|_{U_i} \rightarrow \tilde{\mathcal{O}}|_{U_i}$, see [1]. We put

$$\delta_i(\sum h_j e_j^i + \sum g_j f_j^i) = \sum \sigma_i(h_j) \hat{e}_j^i + \sum \sigma_i(g_j) \hat{f}_j^i, \quad h_j, g_j \in \mathcal{O}.$$

Obviously, δ_i is an isomorphism. We put $\gamma_{ij} := \sigma_i \circ \sigma_j^{-1}$ and $(g_{ij})_{\gamma_{ij}} := \delta_i \circ \delta_j^{-1}$. Moreover, $(\gamma_{ij}) \in Z^1(\mathcal{U}, \mathcal{A}ut_{(2)}\tilde{\mathcal{O}})$, see [1] for more details. We want to show that

$$((g_{ij})_{\gamma_{ij}}) \in Z^1(\mathcal{U}, \mathcal{A}ut_{(2)(2)}\tilde{\mathcal{E}}).$$

Let us take $v \in \tilde{\mathcal{E}}|_{U_j}$, $v = \sum h_k \hat{e}_k^i + \sum g_k \hat{f}_k^i$, $h_j, g_j \in \tilde{\mathcal{O}}$. Then by definition we have

$$\delta_j^{-1}(v) = \sum \sigma_j^{-1}(h_k) e_k^j + \sum \sigma_j^{-1}(g_k) f_k^j.$$

The transition functions of $\tilde{\mathcal{E}}$ may be expressed in $U_i \cap U_j$ as follows:

$$e_k^j = \sum a_s^k e_s^i + \sum b_s^k f_s^i, \quad f_k^j = \sum c_s^k e_s^i + \sum d_s^k f_s^i, \quad a_s^k, d_s^k \in \mathcal{O}_{\bar{0}}, \quad b_s^k, c_s^k \in \mathcal{O}_{\bar{1}}.$$

Further,

$$\alpha(e_k^j) = \hat{e}_k^j = \sum \pi(a_s^k) \hat{e}_s^i, \quad \beta(f_k^j) = \hat{f}_k^j = \sum \tau(c_s^k) \hat{e}_s^i + \sum \pi(d_s^k) \hat{f}_s^i.$$

We have

$$\begin{aligned} \delta_j \circ \delta_j^{-1}(v) &= \sum_k \gamma_{ij}(h_k) (\sum_s \sigma_i(a_s^k) \hat{e}_s^i + \sum_r \sigma_i(b_r^k) \hat{f}_r^i) + \\ &\quad \sum_k \gamma_{ij}(g_k) (\sum_s \sigma_i(c_s^k) \hat{e}_s^i + \sum_s \sigma_i(d_s^k) \hat{f}_s^i) = \\ &= \sum_k h_k (\sum_s \pi(a_s^k) \hat{e}_s^i) + \sum_k g_k (\sum_s \tau(c_s^k) \hat{e}_s^i + \\ &\quad \sum_s \pi(d_s^k) \hat{f}_s^i) \bmod \tilde{\mathcal{E}}_{(2)} = v \bmod \tilde{\mathcal{E}}_{(2)}. \end{aligned}$$

The rest of the proof is a direct repetition of the proof of Theorem 2 from [3]. \square

2. The spectral sequence

2.1. Quasi-derivations

Quasi-derivations were defined in [3]. Let us briefly recall that construction. Consider a locally free sheaf \mathcal{E} on a supermanifold (M, \mathcal{O}) . An even vector space sheaf morphism $A_\Gamma : \mathcal{E} \rightarrow \mathcal{E}$ is called a *quasi-derivation* if $A_\Gamma(fv) = \Gamma(f)v + fA_\Gamma(v)$, where $f \in \mathcal{O}$, $v \in \mathcal{E}$ and Γ is a certain even super vector field. Denote by $Der\mathcal{E}$ the sheaf of quasi-derivations. It is a sheaf of Lie algebras

with respect to the commutator $[A_\Gamma, B_\Upsilon] := A_\Gamma \circ B_\Upsilon - B_\Upsilon \circ A_\Gamma$. The sheaf $Der\mathcal{E}$ possesses a double filtration

$$\begin{array}{ccccc} Der_{(0)(0)}\mathcal{E} & \supset & Der_{(2)(0)}\mathcal{E} & \supset & \cdots \\ \cup & & \cup & & \\ Der_{(0)(2)}\mathcal{E} & \supset & Der_{(2)(2)}\mathcal{E} & \supset & \cdots \\ \vdots & & \vdots & & \end{array}$$

where

$$Der_{(p)(q)}\mathcal{E} := \{A_\Gamma \in Der\mathcal{E} \mid A_\Gamma(\mathcal{E}_{(r)}) \subset \mathcal{E}_{(r+p)}, \Gamma(\mathcal{J}^s) \subset \mathcal{J}^{s+q}, r, s \in \mathbb{Z}\},$$

where $p, q \geq 0$. The map defined by the usual exponential series

$$\exp : Der_{(p)(q)}\mathcal{E} \rightarrow Aut_{(p)(q)}\mathcal{E}, \quad p, q \geq 2,$$

is an isomorphism of sheaves of sets, because operators from $Der_{(p)(q)}\mathcal{E}$, $p, q \geq 2$, are nilpotent. The inverse map is given by the logarithmic series. Define the vector space subsheaf $Der_{k,k}\tilde{\mathcal{E}}$ of $Der_{(k)(k)}\tilde{\mathcal{E}}$ for $k \geq 0$ by

$$Der_{k,k}\tilde{\mathcal{E}} := \{A_\Gamma \in Der_{(k)(k)}\tilde{\mathcal{E}} \mid A_\Gamma(\tilde{\mathcal{E}}_r) \subset \tilde{\mathcal{E}}_{r+k}, \Gamma(\tilde{\mathcal{O}}_s) \subset \tilde{\mathcal{O}}_{s+k}, r, s \in \mathbb{Z}\}.$$

For an even $k \geq 2$, define a map

$$\mu_k : Aut_{(k)(2)}\tilde{\mathcal{E}} \rightarrow Der_{k,k}\tilde{\mathcal{E}}, \quad \mu_k(a_\gamma) = \bigoplus_q \text{pr}_{q+k} \circ A_\Gamma \circ \text{pr}_q,$$

where $a_\gamma = \exp(A_\Gamma)$ and $\text{pr}_k : \tilde{\mathcal{E}} \rightarrow \tilde{\mathcal{E}}_k$ is the natural projection. The kernel of this map is $Aut_{(k+2)(2)}\tilde{\mathcal{E}}$. Moreover, the sequence

$$0 \rightarrow Aut_{(k+2)(2)}\tilde{\mathcal{E}} \longrightarrow Aut_{(k)(2)}\tilde{\mathcal{E}} \xrightarrow{\mu_k} Der_{k,k}\tilde{\mathcal{E}} \rightarrow 0,$$

where $k \geq 2$ is even, is exact. Denoting by $H_{(k)}(\tilde{\mathcal{E}})$ the image of the natural mapping $H^1(M, Aut_{(k)(2)}\tilde{\mathcal{E}}) \rightarrow H^1(M, Aut_{(2)(2)}\tilde{\mathcal{E}})$, we get the filtration

$$H^1(M, Aut_{(2)}\tilde{\mathcal{E}}) = H_{(2)}(\tilde{\mathcal{E}}) \supset H_{(4)}(\tilde{\mathcal{E}}) \supset \dots$$

Take $a_\gamma \in H_{(2)}(\tilde{\mathcal{E}})$. We define the *order* of a_γ to be the maximal number k such that $a_\gamma \in H_{(k)}(\tilde{\mathcal{E}})$. The *order of a locally free sheaf* \mathcal{E} of \mathcal{O} -modules on a supermanifold (M, \mathcal{O}_M) is by definition the order of the corresponding cohomology class.

2.2. The spectral sequence.

A spectral sequence connecting the cohomology with values in the tangent sheaf \mathcal{T} of a supermanifold (M, \mathcal{O}) with the cohomology with values in the tangent sheaf \mathcal{T}_{gr} of the retract $(M, \tilde{\mathcal{O}})$ was constructed in [2]. Here we use similar ideas to construct a new spectral sequence connecting the cohomology with values in a locally free sheaf \mathcal{E} on a supermanifold (M, \mathcal{O}) with the cohomology with values in the locally free sheaf $\tilde{\mathcal{E}}$ on $(M, \tilde{\mathcal{O}})$. Note that our spectral sequence is not a generalization of the spectral sequence obtained in [2] because \mathcal{T}_{gr} is not in general isomorphic to $\tilde{\mathcal{T}}$.

Let \mathcal{E} be a locally free sheaf on a supermanifold (M, \mathcal{O}) of dimension $n|m$. We fix an open Stein covering $\mathfrak{U} = (U_i)_{i \in I}$ of M and consider the

corresponding Čech cochain complex $C^*(\mathfrak{U}, \mathcal{E}) = \bigoplus_{p \geq 0} C^p(\mathfrak{U}, \mathcal{E})$. The \mathbb{Z}_2 -grading of \mathcal{E} gives rise to the \mathbb{Z}_2 -gradings in $C^*(\mathfrak{U}, \mathcal{E})$ and $H^*(M, \mathcal{E})$ given by

$$\begin{aligned} C_{\bar{0}}(\mathfrak{U}, \mathcal{E}) &= \bigoplus_{q \geq 0} C^{2q}(\mathfrak{U}, \mathcal{E}_{\bar{0}}) \oplus \bigoplus_{q \geq 0} C^{2q+1}(\mathfrak{U}, \mathcal{E}_{\bar{1}}), \\ C_{\bar{1}}(\mathfrak{U}, \mathcal{E}) &= \bigoplus_{q \geq 0} C^{2q}(\mathfrak{U}, \mathcal{E}_{\bar{1}}) \oplus \bigoplus_{q \geq 0} C^{2q+1}(\mathfrak{U}, \mathcal{E}_{\bar{0}}). \\ H_{\bar{0}}(M, \mathcal{E}) &= \bigoplus_{q \geq 0} H^{2q}(M, \mathcal{E}_{\bar{0}}) \oplus \bigoplus_{q \geq 0} H^{2q+1}(M, \mathcal{E}_{\bar{1}}), \\ H_{\bar{1}}(M, \mathcal{E}) &= \bigoplus_{q \geq 0} H^{2q}(M, \mathcal{E}_{\bar{1}}) \oplus \bigoplus_{q \geq 0} H^{2q+1}(M, \mathcal{E}_{\bar{0}}). \end{aligned} \tag{2.1}$$

The filtration (1.2) for \mathcal{E} gives rise to the filtration

$$C^*(\mathfrak{U}, \mathcal{E}) = C_{(0)} \supset \dots \supset C_{(p)} \supset \dots \supset C_{(m+2)} = 0 \tag{2.2}$$

of this complex by the subcomplexes

$$C_{(p)} = C^*(\mathfrak{U}, \mathcal{E}_{(p)}).$$

Denoting by $H(M, \mathcal{E})_{(p)}$ the image of the natural mapping $H^*(M, \mathcal{E}_{(p)}) \rightarrow H^*(M, \mathcal{E})$, we get the filtration

$$H^*(M, \mathcal{E}) = H(M, \mathcal{E})_{(0)} \supset \dots \supset H(M, \mathcal{E})_{(p)} \supset \dots \tag{2.3}$$

Denote by $\text{gr } H^*(M, \mathcal{E})$ the bigraded group associated with the filtration (2.3); its bigrading is given by

$$\text{gr } H^*(M, \mathcal{E}) = \bigoplus_{p, q \geq 0} \text{gr}_p H^q(M, \mathcal{E}).$$

By the (more general) Leray procedure, we get a spectral sequence of bigraded groups E_r converging to $E_\infty \simeq \text{gr } H^*(M, \mathcal{E})$. For convenience of the reader, we recall the main definitions here.

For any $p, r \geq 0$, define the vector spaces

$$C_r^p = \{c \in C_{(p)} \mid dc \in C_{(p+r)}\}.$$

Then, for a fixed p , we have

$$C_{(p)} = C_0^p \supset \dots \supset C_r^p \supset C_{r+1}^p \supset \dots$$

The r -th term of the spectral sequence is defined by

$$E_r = \bigoplus_{p=0}^m E_r^p, \quad r \geq 0, \quad \text{where } E_r^p = C_r^p / C_{r-1}^{p+1} + dC_{r-1}^{p-r+1}.$$

Since $d(C_r^p) \subset C_r^{p+r}$, d induces a derivation d_r of E_r of degree r such that $d_r^2 = 0$. Then E_{r+1} is naturally isomorphic to the homology algebra $H(E_r, d_r)$. The \mathbb{Z}_2 -grading (2.1) in $C^*(\mathfrak{U}, \mathcal{E})$ gives rise to certain \mathbb{Z}_2 -gradings in C_r^p and E_r^p , turning E_r into a superspace. Clearly, the coboundary operator d on $C^*(\mathfrak{U}, \mathcal{E})$ is odd. It follows that the coboundary d_r is odd for any $r \geq 0$.

The superspaces E_r are also endowed with a second \mathbb{Z} -grading. Namely, for any $q \in \mathbb{Z}$, set

$$C_r^{p,q} = C_r^p \cap C^{p+q}(\mathfrak{U}, \mathcal{E}), \quad E_r^{p,q} = C_r^{p,q} / C_{r-1}^{p+1,q-1} + dC_{r-1}^{p-r+1,q+r-2}.$$

Then

$$E_r = \bigoplus_{p,q} E_r^{p,q} \text{ and } d_r(E_r^{p,q}) \subset E_r^{p+r,q-r+1} \text{ for any } r, p, q. \quad (2.4)$$

Further, for a fixed q , we have $d(C_r^{p,q}) = 0$ for all $p \geq 0$ and all $r \geq m+2$. This implies that the natural homomorphism $E_r^{p,q} \rightarrow E_{r+1}^{p,q}$ is an isomorphism for all p and $r \geq r_0 = m+2$. Setting $E_\infty^{p,q} = E_{r_0}^{p,q}$, we get the bigraded superspace

$$E_\infty = \bigoplus_{p,q} E_\infty^{p,q}.$$

Lemma 2.1. *The first two terms of the spectral sequence (E_r) can be identified with the following bigraded spaces:*

$$E_0 = C^*(\mathfrak{U}, \tilde{\mathcal{E}}), \quad E_1 = E_2 = H^*(M, \tilde{\mathcal{E}}).$$

More precisely,

$$E_0^{p,q} = C^{p+q}(\mathfrak{U}, \tilde{\mathcal{E}}_p), \quad E_1^{p,q} = E_2^{p,q} = H^{p+q}(M, \tilde{\mathcal{E}}_p).$$

We have $d_{2k+1} = 0$ and, hence, $E_{2k+1} = E_{2k+2}$ for all $k \geq 0$.

Proof. The proof is similar to the proof of Proposition 3 in [2]. \square

Lemma 2.2. *There is the following identification of bigraded algebras:*

$$E_\infty = \text{gr } H^*(M, \mathcal{E}), \text{ where } E_\infty^{p,q} = \text{gr}_p H^{p+q}(M, \mathcal{E}).$$

If M is compact, then $\dim H^k(M, \mathcal{E}) = \sum_{p+q=k} \dim E_\infty^{p,q}$.

Proof. The proof is a direct repetition of the proof of Proposition 4 in [2]. \square

Now we prove our main result concerning the first non-zero coboundary operators among d_2, d_4, \dots . Assume that the isomorphisms of sheaves $\delta_i : \mathcal{E}|U_i \rightarrow \tilde{\mathcal{E}}|U_i$ from Theorem 1.1 are defined for each $i \in I$. By Theorem 1.1, a locally free sheaf of \mathcal{O} -modules \mathcal{E} on M corresponds to the cohomology class a_γ of the 1-cocycle $((a_\gamma)_{ij}) \in Z^1(\mathfrak{U}, \mathcal{A}ut_{(2)(2)}\tilde{\mathcal{E}})$, where $(a_\gamma)_{ij} = \delta_i \circ \delta_j^{-1}$. If the order of $(a_\gamma)_{ij}$ is equal to k , then we may choose $\delta_i, i \in I$, in such a way that $((a_\gamma)_{ij}) \in Z^1(\mathfrak{U}, \mathcal{A}ut_{(k)(2)}\tilde{\mathcal{E}})$. We can write $a_\gamma = \exp A_\Gamma$, where $A_\Gamma \in C^1(\mathfrak{U}, \text{Der}_{(k)(2)}\tilde{\mathcal{E}})$.

We will identify the superspaces (E_0, d_0) and $(C^*(\mathfrak{U}, \tilde{\mathcal{E}}), d)$ via the isomorphism of Lemma 2.1. Clearly, $\delta_i : \mathcal{E}_{(p)}|U_i \rightarrow \tilde{\mathcal{E}}_{(p)}|U_i = \sum_{r \geq p} \tilde{\mathcal{E}}_r|U_i$ is an isomorphism of sheaves for all $i \in I, p \geq 0$. These local sheaf isomorphisms permit us to define an isomorphism of graded cochain groups

$$\psi : C^*(\mathfrak{U}, \mathcal{E}) \rightarrow C^*(\mathfrak{U}, \tilde{\mathcal{E}})$$

such that

$$\psi : C^*(\mathfrak{U}, \mathcal{E}_{(p)}) \rightarrow C^*(\mathfrak{U}, (\tilde{\mathcal{E}})_{(p)}), \quad p \geq 0.$$

We put

$$\psi(c)_{i_0 \dots i_q} = \delta_{i_0}(c_{i_0 \dots i_q})$$

for any (i_0, \dots, i_q) such that $U_{i_0} \cap \dots \cap U_{i_q} \neq \emptyset$. Note that ψ is not an isomorphism of complexes. Nevertheless, we can explicitly express the coboundary d of the complex $C^*(\mathfrak{U}, \mathcal{E})$ by means of d_0 and a_γ .

The following theorem permits to calculate the spectral sequence (E_r) whenever d_0 and the cochain a_γ are known. It also describes certain coboundary operators d_r , $r \geq 1$.

Theorem 2.3. *For any $c \in C^*(\mathfrak{U}, \tilde{\mathcal{E}}_q) = E_0^q$, we have*

$$(\psi(d\psi^{-1}(c)))_{i_0 \dots i_{q+1}} = (d_0 c)_{i_0 \dots i_{q+1}} + ((a_\gamma)_{i_0 i_1} - \text{id})(c_{i_1 \dots i_{q+1}}).$$

Suppose that the locally free sheaf of \mathcal{O} -modules \mathcal{E} on M has order k and denote by a_γ the cohomology class corresponding to \mathcal{E} by Theorem 1.1. Then $d_r = 0$ for $r = 1, \dots, k-1$, and $d_k = \mu_k(a_\gamma)$.

Proof. The proof is similar to the proof of Theorem 7 in [3]. □

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